Appendix 1: Penumbra Transit Duration, Preliminary Calculation

A1.1 Introduction

The minimum duration possible of Earth penumbra transit for a given orbital height (assuming a circular orbit) would be produced by a radial trajectory through the centre of the umbra, whereas the maximum duration of penumbra passage would be that produced by any trajectory tangential to the umbra (Figure 1). Calculation of the arclength through the penumbra of both types of trajectory for given orbital heights and dividing by the mean orbital motion (rad s^{-1}) for that height gives the duration of penumbra passage. These calculations were based on an Earth radius of 6378 km, i.e. the "expanded Earth" concept mentioned in chapter 3 was not used.

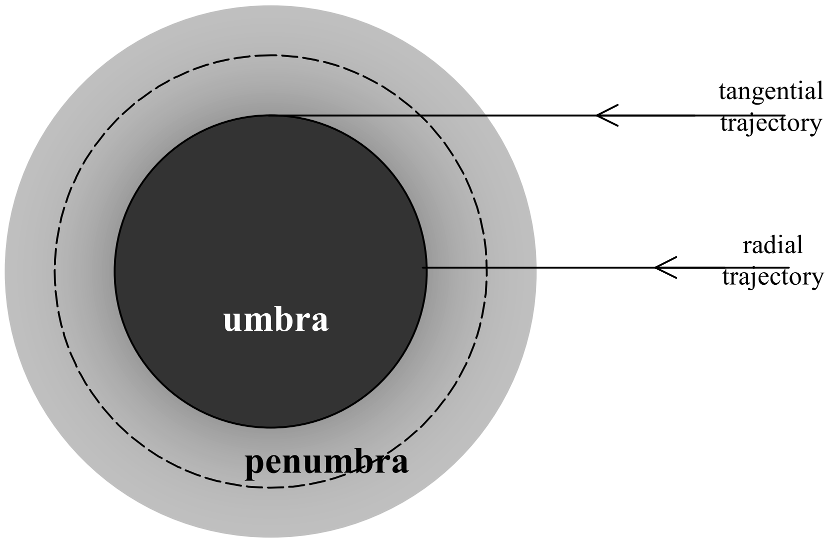

Figure 1: Tangential and radial trajectories through the Earth's shadow. The trajectory at a tangent to the umbra is the longest path within the penumbra. The dashed circle illustrates the size of the cross-section of a cylindrical shadow for comparison, lying between the true umbra and penumbra (qualitative only).

A1.2 Radial Transit

The case of radial transit is treated as a coplanar system. With reference to Figure 2, the shadow axis is denoted by the line OC, the terminator for the penumbra is shown as T_u, and that for the penumbra is T_p. The edges of the umbra and penumbra are shown as the lines T_uC and T_pX respectively. A circular orbit of height h cuts the umbra and penumbra at P and Q respectively. The angle T_uOT_p is equal to the sum of the umbra and penumbra half angles θ_u and θ_p and is equal to the angle POQ. Mean motion (mean angular velocity) for circular orbits at height h is given by:

$$n = \sqrt{\frac{G M_e}{(R_e + h)^3}} \quad \text{rad s}^{-1}, \tag{A1.1}$$

where G = gravitational constant, and M_e = Mass of Earth. Therefore the time to cross the penumbra radially is given by $\angle$ POQ / n.

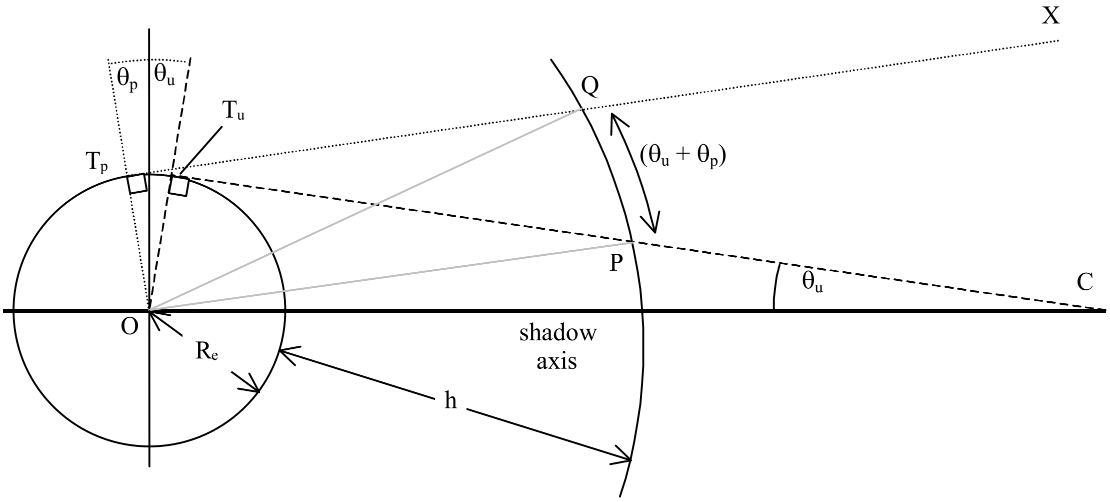

Figure 2: Geometry for minimum penumbra transit time (radial trajectory).

A1.3 Tangential Trajectory

For the maximum duration path across the penumbra to the umbra, the tangential path length is solved in part using spherical trigonometry. With reference to Figure 3, the tangential path x enters the penumbra at A and touches the umbra at B. By definition the angle ABC is a right angle, making triangle ABC a right spherical triangle, the standard solution of which gives the arclength x as:

$$x = \cos^{-1}\left[\frac{\cos r_p}{\cos r_u}\right], \tag{A1.2}$$

where r_p = angular radius of the penumbra, and r_u is that of the umbra. The values of r_p and r_u can be taken from Figure 2 where angle OPC is obtained from the sine rule to be:

$$\angle OPC = \sin^{-1}\left[\sin\theta_c \cdot \frac{D}{R_e + h}\right], \tag{A1.3}$$

thus in completing the triangle OPC, the angle POC, or r_u, is given by

$$r_u = \pi - \theta_c - \sin^{-1}\left[\sin\theta_c \cdot \frac{D}{R_e + h}\right]. \qquad \text{(A1.4)}$$

Since from figure 2, $\angle$ QOC $=$ $\angle$ POC $+ 2\,\theta_c$, then $r_p = r_u + 2\,\theta_c$.

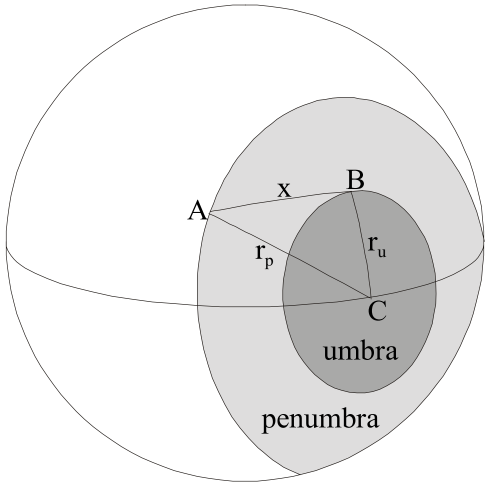

Figure 3 Geometry governing tangential passage of penumbra; i.e. such that the trajectory is a tangent to the umbra. Finding x gives the maximum duration of penumbral transit involving contact with the umbra. All sides of the triangle are great circle arcs. See text for details.

The graph shows that at the very least (radial trajectory), transit times are large compared to the nominal timestep size for observing. For elliptical orbits transit times would be even longer near apogee due to the lower orbital velocity at the same height. Therefore during eclipse the orbiting body will spend many time increments within the penumbra, so the illumination environment within the penumbra is not negligible and must be calculated as part of the Skyplot program.

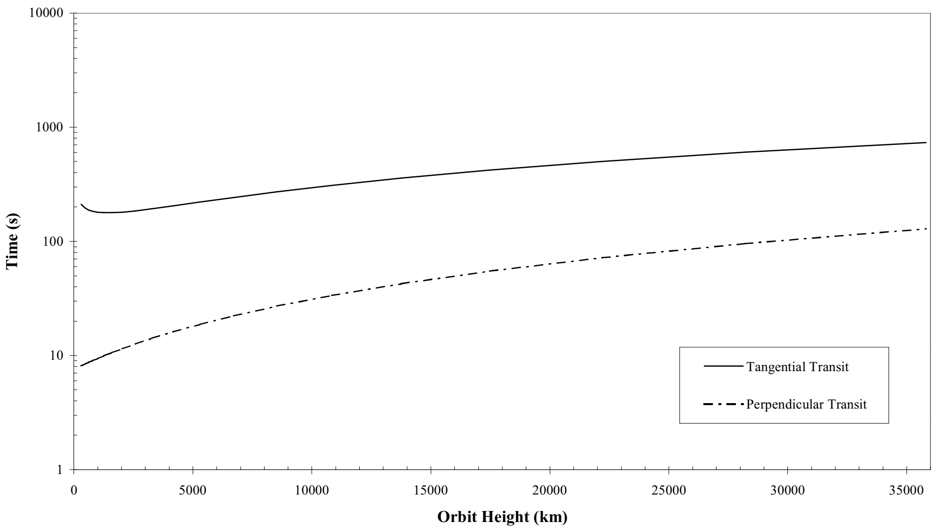

Figure 4: Maximum and minimum penumbra transit times for a circular orbit.